Calculus Integrals Quick Reference

A concise cheat sheet covering fundamental integration rules, techniques, common integrals, and applications of calculus.

Introduction to Integrals & Basic Rules

Antiderivatives and Indefinite Integrals

Antiderivative: A function $F(x)$ is an antiderivative of $f(x)$ if $F'(x) = f(x)$.

Indefinite Integral: The set of all antiderivatives of $f(x)$ is denoted by $\int f(x)dx = F(x) + C$, where $F(x)$ is *any* antiderivative and C is the constant of integration.

$\int dx = x + C$
$\int c \cdot f(x)dx = c \int f(x)dx$
$\int (f(x) \pm g(x))dx = \int f(x)dx \pm \int g(x)dx$

Why the + C?

Different functions can have the same derivative. For example, x^2, $x^2 + 5$, and $x^2 - 10$ all have a derivative of $2x$. The constant C represents this arbitrary constant difference between any two antiderivatives.

Tip: Always remember to add the +C for indefinite integrals!

Basic Power and Constant Rules

Power Rule for Integration ($n \neq -1$)	$\int x^n dx = \frac{x^{n+1}}{n+1} + C$
Example: $\int x^3 dx$	$\int x^3 dx = \frac{x^{3+1}}{3+1} + C = \frac{x^4}{4} + C$
Integration of a Constant	$\int c dx = cx + C$
Example: $\int 5dx$	$\int 5dx = 5x + C$
Integration of $1/x$	$\int \frac{1}{x} dx = \ln\|x\| + C (Note the absolute value for x < 0)$
Example: $\int \frac{2}{x} dx$	$\int \frac{2}{x} dx = 2 \int \frac{1}{x} dx = 2 \ln\|x\| + C$

Common Integrals (Trigonometric)

Integral of $\sin(x)$	$\int \sin(x)dx = -\cos(x) + C$
Integral of $\cos(x)$	$\int \cos(x)dx = \sin(x) + C$
Integral of $\tan(x)$	$\int \tan(x)dx = -\ln\|\cos(x)\| + C = \ln\|\sec(x)\| + C$
Integral of $\sec(x)$	$\int \sec(x)dx = \ln\|\sec(x) + \tan(x)\| + C$
Integral of $\sec^2(x)$	$\int \sec^2(x)dx = \tan(x) + C$
Integral of $\csc(x)\cot(x)$	$\int \csc(x)\cot(x)dx = -\csc(x) + C$

Techniques of Integration

U-Substitution (Change of Variables)

Concept: Reverse of the Chain Rule. Used when you have an integrand of the form $f(g(x))g'(x)dx$.

Steps:

1. Choose a substitution $u = g(x)$.
2. Compute the differential $du = g'(x)dx$.
3. Rewrite the integral entirely in terms of u and du.
4. Evaluate the integral with respect to u.
5. Substitute back $u = g(x)$ to express the result in terms of the original variable.

Example: $\int x \cos(x^2)dx$

1. Let $u = x^2$.
2. $du = 2xdx$, so $xdx = \frac{1}{2}du$.
3. $\int \cos(u)\frac{1}{2}du = \frac{1}{2}\int \cos(u)du$.
4. $\frac{1}{2}\sin(u) + C$.
5. Substitute back: $\frac{1}{2}\sin(x^2) + C$.

Tip: Look for an inner function ($g(x)$) whose derivative ($g'(x)$) is also present in the integrand (up to a constant factor).

Integration by Parts

Concept: Reverse of the Product Rule. Used for integrals of products of functions.

$\int u\,dv = uv - \int v\,du$

Formula: $\int u\,dv = uv - \int v\,du$

Mnemonic (LIATE): A guideline for choosing u in $\int u\,dv$. Choose u as the function type appearing first in this list:

1. **L**ogarithmic functions ($\ln x$, $\log_b x$)
2. **I**nverse trigonometric functions ($\arcsin x$, $\arctan x$, etc.)
3. **A**lgebraic functions (x^n, polynomials)
4. **T**rigonometric functions ($\sin x$, $\cos x$, etc.)
5. **E**xponential functions (e^x, a^x)

Example: $\int xe^x dx$

LIATE suggests $u = x$ (Algebraic) and $dv = e^x dx$ (Exponential).

1. $u = x \implies du = dx$
2. $dv = e^x dx \implies v = \int e^x dx = e^x$

Now apply the formula:
$\int xe^x dx = xe^x - \int e^x dx$
$= xe^x - e^x + C$
$= e^x(x-1) + C$

Example 2: $\int \ln x dx$

Let $u = \ln x$ and $dv = dx$.

1. $u = \ln x \implies du = \frac{1}{x} dx$
2. $dv = dx \implies v = x$

Apply the formula:
$\int \ln x dx = (\ln x)(x) - \int x \frac{1}{x} dx$
$= x \ln x - \int 1 dx$
$= x \ln x - x + C$

Trigonometric Integrals

Involving Powers of $\sin(x)$ and $\cos(x)$: $\int \sin^m(x)\cos^n(x)dx$

- **Case 1: n is odd.** Save one $\cos(x)$ factor and use $\cos^2(x) = 1 - \sin^2(x)$ for the remaining factors. Let $u = \sin(x)$, $du = \cos(x)dx$.
  - Example: $\int \sin^2(x)\cos^3(x)dx = \int \sin^2(x)(1-\sin^2(x))\cos(x)dx$. Let $u = \sin(x)$, $du = \cos(x)dx$.
    $\int u^2(1-u^2)du = \int(u^2 - u^4)du = \frac{u^3}{3} - \frac{u^5}{5} + C = \frac{\sin^3(x)}{3} - \frac{\sin^5(x)}{5} + C$.
- **Case 2: m is odd.** Save one $\sin(x)$ factor and use $\sin^2(x) = 1 - \cos^2(x)$ for the remaining factors. Let $u = \cos(x)$, $du = -\sin(x)dx$.
  - Example: $\int \sin^3(x)\cos^2(x)dx$. Let $u = \cos(x)$, $du = -\sin(x)dx$. $\int(1-u^2)u^2(-du) = \int(u^4 - u^2)du = \frac{u^5}{5} - \frac{u^3}{3} + C = \frac{\cos^5(x)}{5} - \frac{\cos^3(x)}{3} + C$.
- **Case 3: Both m and n are even.** Use half-angle identities: $\sin^2(x) = \frac{1-\cos(2x)}{2}$, $\cos^2(x) = \frac{1+\cos(2x)}{2}$.
  - Example: $\int \cos^2(x)dx = \int \frac{1+\cos(2x)}{2}dx = \frac{1}{2}\int dx + \frac{1}{2}\int \cos(2x)dx = \frac{x}{2} + \frac{1}{4}\sin(2x) + C$.

Involving Powers of $\tan(x)$ and $\sec(x)$: $\int \tan^m(x)\sec^n(x)dx$

- **Case 1: n is even ($n = 2k, k \geq 1$).** Save a $\sec^2(x)$ factor and use $\sec^2(x) = 1 + \tan^2(x)$ for the remaining factors. Let $u = \tan(x)$, $du = \sec^2(x)dx$.
  - Example: $\int \tan^3(x)\sec^4(x)dx = \int \tan^3(x)(1+\tan^2(x))\sec^2(x)dx$. Let $u = \tan(x)$, $du = \sec^2(x)dx$.
    $\int u^3(1+u^2)du = \int(u^3 + u^5)du = \frac{u^4}{4} + \frac{u^6}{6} + C = \frac{\tan^4(x)}{4} + \frac{\tan^6(x)}{6} + C$.
- **Case 2: m is odd ($m = 2k+1$).** Save a $\sec(x)\tan(x)$ factor and use $\tan^2(x) = \sec^2(x) - 1$ for the remaining factors. Let $u = \sec(x)$, $du = \sec(x)\tan(x)dx$.
  - Example: $\int \tan^3(x)\sec(x)dx = \int(\sec^2(x) - 1)\sec(x)\tan(x)dx$. Let $u = \sec(x)$, $du = \sec(x)\tan(x)dx$.
    $\int(u^2 - 1)du = \frac{u^3}{3} - u + C = \frac{\sec^3(x)}{3} - \sec(x) + C$.
- **Other Cases:** Often require reduction formulas or special tricks.

Definite Integrals and Applications

Fundamental Theorem of Calculus (FTC)

FTC Part 1: If f is continuous on $[a, b]$, then the function $g(x)$ defined by $g(x) = \int_a^x f(t)dt$ is continuous on $[a, b]$ and differentiable on (a, b), and $g'(x) = f(x)$.

$\frac{d}{dx} \int_a^x f(t)dt = f(x)$

Example: $\frac{d}{dx} \int_1^x \sin(t^2)dt = \sin(x^2)$

With Chain Rule: $\frac{d}{dx} \int_a^{g(x)} f(t)dt = f(g(x)) \cdot g'(x)$

Example: $\frac{d}{dx} \int_0^{x^3} e^t dt = e^{x^3} \cdot \frac{d}{dx}(x^3) = e^{x^3} \cdot 3x^2$

FTC Part 2: If f is continuous on $[a, b]$, then $\int_a^b f(x)dx = F(b) - F(a)$, where F is any antiderivative of f, i.e., $F'(x) = f(x)$.

$\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a)$

Evaluating Definite Integrals

Steps:

1. Find the indefinite integral (antiderivative) $F(x)$ of the integrand $f(x)$. Don't need the $+C$ here as it cancels out.
2. Evaluate $F(x)$ at the upper limit (b) and the lower limit (a).
3. Subtract the value at the lower limit from the value at the upper limit: $F(b) - F(a)$.

Example: $\int_1^2 x^2 dx$

1. Antiderivative of x^2 is $\frac{x^3}{3}$. So $F(x) = \frac{x^3}{3}$.
2. Evaluate at limits: $F(2) = \frac{2^3}{3} = \frac{8}{3}$, $F(1) = \frac{1^3}{3} = \frac{1}{3}$.
3. Subtract: $F(2) - F(1) = \frac{8}{3} - \frac{1}{3} = \frac{7}{3}$.

Tip: If using U-substitution for a definite integral, you can change the limits of integration when you change the variable. If $u = g(x)$, the new limits become $g(a)$ and $g(b)$. This avoids substituting back to x before evaluating.

Applications: Area Between Curves

Concept: The area between two curves $y = f(x)$ and $y = g(x)$ from $x = a$ to $x = b$, where $f(x) \geq g(x)$ on $[a, b]$.

Formula: Area $= \int_a^b [f(x) - g(x)]dx$

- $f(x)$ is the **upper** curve, $g(x)$ is the **lower** curve.
- If the upper/lower curve changes within the interval, you must split the integral into multiple integrals.
- Alternatively, integrate with respect to y: Area $= \int_c^d [h(y) - k(y)]dy$, where $x = h(y)$ is the right curve and $x = k(y)$ is the left curve, from $y = c$ to $y = d$.

Example: Find the area bounded by $y = x^2$ and $y = x$ for $x \geq 0$.

1. Find intersection points: $x^2 = x \implies x^2 - x = 0 \implies x(x-1) = 0 \implies x = 0, x = 1$. So $a = 0, b = 1$.
2. On $[0, 1]$, $y = x$ is the upper curve ($f(x) = x$) and $y = x^2$ is the lower curve ($g(x) = x^2$).
3. Area $= \int_0^1 [x - x^2]dx = [\frac{x^2}{2} - \frac{x^3}{3}]_0^1 = (\frac{1^2}{2} - \frac{1^3}{3}) - (\frac{0^2}{2} - \frac{0^3}{3}) = (\frac{1}{2} - \frac{1}{3}) - 0 = \frac{3-2}{6} = \frac{1}{6}$.

Concept: Finding the volume of a solid of revolution formed by rotating a region about an axis.

Disk Method: Used when the region is adjacent to the axis of revolution, creating solid disks.

- Rotation about x-axis: $V = \int_a^b \pi[f(x)]^2 dx$
- Rotation about y-axis: $V = \int_c^d \pi[g(y)]^2 dy$ (where $x = g(y)$)

Washer Method: Used when there is a space between the region and the axis of revolution, creating washers (disks with holes).

- Rotation about x-axis: $V = \int_a^b \pi([R(x)]^2 - [r(x)]^2) dx$, where $R(x)$ is the outer radius and $r(x)$ is the inner radius.
- Rotation about y-axis: $V = \int_c^d \pi([R(y)]^2 - [r(y)]^2) dy$, where $R(y)$ is the outer radius and $r(y)$ is the inner radius.

Example (Disk): Volume of solid formed by rotating the region under $y = \sqrt{x}$ from $x = 0$ to $x = 4$ about the x-axis.

- $a = 0, b = 4, f(x) = \sqrt{x}$.
- $V = \int_0^4 \pi(\sqrt{x})^2 dx = \pi \int_0^4 x dx = \pi[\frac{x^2}{2}]_0^4 = \pi(\frac{4^2}{2} - \frac{0^2}{2}) = \pi(8 - 0) = 8\pi$.

Example (Washer): Volume of solid formed by rotating the region bounded by $y = x^2$ and $y = x$ $(x \geq 0)$ about the x-axis.

- Intersection points at $x = 0, x = 1$. So $a = 0, b = 1$.
- Outer radius $R(x)$ is the function farther from the axis, $y = x$. So $R(x) = x$.
- Inner radius $r(x)$ is the function closer to the axis, $y = x^2$. So $r(x) = x^2$.
- $V = \int_0^1 \pi[x^2 - (x^2)^2] dx = \pi \int_0^1 [x^2 - x^4] dx = \pi[\frac{x^3}{3} - \frac{x^5}{5}]_0^1 = \pi((\frac{1^3}{3} - \frac{1^5}{5}) - (0 - 0)) = \pi(\frac{1}{3} - \frac{1}{5}) = \pi(\frac{5-3}{15}) = \frac{2\pi}{15}$.