

Divisibility and Primes

Basic Divisibility

Divisibility Notation	$a \mid b$ means 'a divides b', i.e., there exists an integer k such that $b = ak$.
Divisor Properties	If $a \mid b$ and $a \mid c$, then $a \mid (bx + cy)$ for any integers x, y.
Transitivity of Divisibility	If $a \mid b$ and $b \mid c$, then $a \mid c$.
Divisibility by a Product	If $a \mid c$, $b \mid c$ and $\gcd(a, b) = 1$, then $ab \mid c$.
Euclidean Algorithm	Efficiently computes the greatest common divisor (GCD) of two integers.
GCD Definition	$\gcd(a, b)$ is the largest positive integer that divides both a and b.

Prime Numbers

Prime Number Definition	A prime number is a natural number greater than 1 that has no positive divisors other than 1 and itself.
Fundamental Theorem of Arithmetic	Every integer greater than 1 can be uniquely represented as a product of prime numbers, up to the order of the factors.
Prime Factorization	Expressing a number as a product of its prime factors (e.g., $12 = 2^2 \cdot 3$).
Infinitude of Primes	There are infinitely many prime numbers.
Mersenne Primes	Primes of the form $2^p - 1$, where p is also prime.
Twin Primes	Pairs of primes that differ by 2 (e.g., 3 and 5, 5 and 7).

Congruences

Modular Arithmetic

Congruence Notation	$a \equiv b \pmod m$ means 'a is congruent to b modulo m', i.e., $m \mid (a - b)$.
Properties of Congruences	If $a \equiv b \pmod m$ and $c \equiv d \pmod m$, then: <ul style="list-style-type: none">$a + c \equiv b + d \pmod m$$a - c \equiv b - d \pmod m$$ac \equiv bd \pmod m$
Cancellation	If $ac \equiv bc \pmod m$ and $\gcd(c, m) = 1$, then $a \equiv b \pmod m$.
Linear Congruences	An equation of the form $ax \equiv b \pmod m$.
Solving Linear Congruences	A solution exists if and only if $\gcd(a, m) \mid b$. If a solution exists, there are $\gcd(a, m)$ solutions modulo m.
Modular Inverse	If $ax \equiv 1 \pmod m$, then x is the modular inverse of a modulo m. Exists if and only if $\gcd(a, m) = 1$.

Important Theorems

Fermat's Little Theorem	If p is prime and $\gcd(a, p) = 1$, then $a^{p-1} \equiv 1 \pmod p$.
Euler's Theorem	If $\gcd(a, m) = 1$, then $a^{\phi(m)} \equiv 1 \pmod m$, where $\phi(m)$ is Euler's totient function.
Euler's Totient Function	$\phi(m)$ counts the number of integers between 1 and m that are relatively prime to m.
Calculating Euler's Totient	If $m = p_1^{k_1} \cdot p_2^{k_2} \cdot \dots \cdot p_n^{k_n}$, then $\phi(m) = m \cdot (1 - 1/p_1) \cdot (1 - 1/p_2) \cdot \dots \cdot (1 - 1/p_n)$.
Chinese Remainder Theorem (CRT)	Given a system of congruences $x \equiv a_1 \pmod{m_1}, x \equiv a_2 \pmod{m_2}, \dots, x \equiv a_n \pmod{m_n}$, where $\gcd(m_i, m_j) = 1$ for all $i \neq j$, there exists a unique solution modulo $M = m_1 \cdot m_2 \cdot \dots \cdot m_n$.
Applying CRT	The CRT provides a method to reconstruct a number from its remainders modulo pairwise coprime moduli.

Diophantine Equations

Linear Diophantine Equations

General Form	$ax + by = c$, where a, b, c are integers, and we seek integer solutions for x and y.
Solvability Condition	A solution exists if and only if $\gcd(a, b) \mid c$.
Finding Solutions	Use the Extended Euclidean Algorithm to find integers x_0, y_0 such that $ax_0 + by_0 = \gcd(a, b)$. If $\gcd(a, b) \mid c$, then $x = x_0 \cdot (c / \gcd(a, b))$ and $y = y_0 \cdot (c / \gcd(a, b))$ is a particular solution.
General Solution	If (x_0, y_0) is a particular solution, then the general solution is given by: $x = x_0 + (b / \gcd(a, b)) \cdot t$ $y = y_0 - (a / \gcd(a, b)) \cdot t$ where t is any integer.
Example	Solve $3x + 6y = 9$. Since $\gcd(3, 6) = 3$ and $3 \mid 9$, a solution exists. From $3x + 6y = 3 \cdot 3$, we simplify to $x + 2y = 3$. A particular solution is $x=3, y=0$. General solution: $x = 3 + 2t, y = -t$.

Pythagorean Triples

Definition	A Pythagorean triple consists of three positive integers a, b, and c, such that $a^2 + b^2 = c^2$.
Primitive Pythagorean Triple	A Pythagorean triple (a, b, c) is primitive if $\gcd(a, b, c) = 1$.
Generating Pythagorean Triples	If m and n are positive integers with $m > n$, $\gcd(m, n) = 1$, and one of m and n is even, then: $a = m^2 - n^2$ $b = 2mn$ $c = m^2 + n^2$ forms a primitive Pythagorean triple.
Example	Let $m = 2$ and $n = 1$. Then: $a = 2^2 - 1^2 = 3$ $b = 2 \cdot 2 \cdot 1 = 4$ $c = 2^2 + 1^2 = 5$ Thus, (3, 4, 5) is a Pythagorean triple.

Arithmetic Functions

Common Arithmetic Functions

Divisor Function ($\sigma(n)$)	$\sigma(n)$ is the sum of all positive divisors of n , including 1 and n itself. $\sigma(n) = \sum_{d n} d$
Number of Divisors ($\tau(n)$ or $d(n)$)	$\tau(n)$ is the number of positive divisors of n . $\tau(n) = \sum_{d n} 1$
Euler's Totient Function ($\varphi(n)$)	$\varphi(n)$ is the number of integers between 1 and n that are relatively prime to n . $\varphi(n) = \{k : 1 \leq k \leq n, \gcd(n, k) = 1\} $
Möbius Function ($\mu(n)$)	$\mu(n)$ is defined as: <ul style="list-style-type: none">0 if n has one or more repeated prime factors.1 if $n = 1$.$(-1)^k$ if n is a product of k distinct primes.
Example: $\sigma(12)$	The divisors of 12 are 1, 2, 3, 4, 6, and 12. Thus, $\sigma(12) = 1 + 2 + 3 + 4 + 6 + 12 = 28$.
Example: $\tau(12)$	The number of divisors of 12 is 6. Thus, $\tau(12) = 6$.

Multiplicativity

Definition	An arithmetic function $f(n)$ is multiplicative if $f(mn) = f(m)f(n)$ whenever $\gcd(m, n) = 1$. It is completely multiplicative if this holds for all m and n .
Examples of Multiplicative Functions	Euler's totient function $\varphi(n)$, the divisor function $\sigma(n)$, and the number of divisors function $\tau(n)$ are multiplicative.
Möbius function	The Möbius function $\mu(n)$ is also multiplicative.
Implications of Multiplicativity	If $f(n)$ is multiplicative and $n = p_1^{k_1} \cdot p_2^{k_2} \cdot \dots \cdot p_r^{k_r}$, then $f(n) = f(p_1^{k_1}) \cdot f(p_2^{k_2}) \cdot \dots \cdot f(p_r^{k_r})$.