CHEAT SHEETS HERO

# Complex Analysis Cheatsheet

A comprehensive cheat sheet covering key concepts, theorems, and formulas in complex analysis, providing a quick reference for students and professionals.

## Complex Numbers and Functions

### Basic Definitions

**Complex Number:** $z = x + iy$, where $x$ and $y$ are real numbers, and $i$ is the imaginary unit ($i^2 = -1$).

**Real Part:** $\text{Re}(z) = x$

**Imaginary Part:** $\text{Im}(z) = y$

**Complex Conjugate:** $\overline{z} = x - iy$

**Modulus:** $|z| = \sqrt{x^2 + y^2}$

**Argument:** $\theta = \arg(z)$, such that $z = |z|e^{i\theta}$

### Complex Functions

**Definition:** A function $f: \mathbb{C} \rightarrow \mathbb{C}$ that maps complex numbers to complex numbers.

**Representation:** $f(z) = u(x, y) + iv(x, y)$, where $u$ and $v$ are real-valued functions.

**Limit:** $\lim_{z \to z_0} f(z) = L$ if for every $\epsilon > 0$, there exists a $\delta > 0$ such that $|f(z) - L| < \epsilon$ whenever $0 < |z - z_0| < \delta$.

**Continuity:** $f(z)$ is continuous at $z_0$ if $\lim_{z \to z_0} f(z) = f(z_0)$.

**Derivative:** $f'(z) = \lim_{h \to 0} \frac{f(z + h) - f(z)}{h}$, if the limit exists.

### Polar Form

**Representation:** $z = r e^{i\theta} = r(\cos \theta + i \sin \theta)$, where $r = |z|$ and $\theta = \arg(z)$

**Multiplication:** $z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)}$

**Division:** $\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}$

**Power:** $z^n = r^n e^{i n \theta}$

**Roots:** $z^{1/n} = r^{1/n} e^{i(\theta + 2\pi k)/n}$, for $k = 0, 1, \ldots, n-1$

## Analytic Functions

### Cauchy-Riemann Equations

**Equations:** $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

**Analyticity:** If $f(z) = u(x, y) + iv(x, y)$ is analytic in a domain D, then the Cauchy-Riemann equations hold in D.

**Sufficient Condition:** If the partial derivatives of $u$ and $v$ are continuous and satisfy the Cauchy-Riemann equations in a domain D, then $f(z)$ is analytic in D.

**Complex Derivative:** $f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y}$

**Harmonic Functions:** If $f(z)$ is analytic, then $u$ and $v$ are harmonic functions, i.e., $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ and $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$.

### Elementary Functions

**Exponential Function:** $e^z = e^{x + iy} = e^x(\cos y + i \sin y)$

**Trigonometric Functions:** $\sin z = \frac{e^{iz} - e^{-iz}}{2i}$, $\cos z = \frac{e^{iz} + e^{-iz}}{2}$

**Hyperbolic Functions:** $\sinh z = \frac{e^z - e^{-z}}{2}$, $\cosh z = \frac{e^z + e^{-z}}{2}$

**Logarithmic Function:** $\log z = \ln |z| + i \arg z$

**Principal Value of Logarithm:** $\text{Log } z = \ln |z| + i \text{Arg } z$, where $-\pi < \text{Arg } z \leq \pi$

**Complex Power:** $z^c = e^{c \log z}$, where $c$ is a complex constant.

## Complex Integration

### Contour Integrals

**Definition:** $\int_C f(z) dz = \int_a^b f(z(t)) z'(t) dt$, where C is a smooth curve parameterized by $z(t)$, $a \leq t \leq b$.

**Properties:** $\int_C [\alpha f(z) + \beta g(z)] dz = \alpha \int_C f(z) dz + \beta \int_C g(z) dz$
$\int_{-C} f(z) dz = -\int_C f(z) dz$

**ML Estimate:** $|\int_C f(z) dz| \leq ML$, where M is an upper bound for $|f(z)|$ on C, and L is the length of C.

### Cauchy's Theorem and Integral Formula

**Cauchy's Theorem:** If $f(z)$ is analytic in a simply connected domain D, then for any closed contour C in D, $\oint_C f(z) dz = 0$.

**Cauchy's Integral Formula:** If $f(z)$ is analytic in a simply connected domain D, and $z_0$ is any point in D inside a closed contour C, then $f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0} dz$.

**Generalized Integral Formula:** $f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}} dz$

## Series and Residues

### Taylor and Laurent Series

**Taylor Series:** If $f(z)$ is analytic in a disk $|z - z_0| < R$, then $f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n$.

**Laurent Series:** If $f(z)$ is analytic in an annulus $r < |z - z_0| < R$, then $f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$, where $a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}} dz$.

## Residue Theorem

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| **Residue:** | The residue of $f(z)$ at an isolated singularity $z_0$ is the coefficient $a_{-1}$ in the Laurent series expansion of $f(z)$ about $z_0$. |
| **Residue Theorem:** | If $f(z)$ is analytic inside and on a closed contour C, except for a finite number of isolated singularities $z_k$ inside C, then $\oint_C f(z) dz = 2\pi i \sum_{k=1}^n \text{Res}(f, z_k)$. |
| **Residue Calculation (Simple Pole):** | If $f(z)$ has a simple pole at $z_0$, then $\text{Res}(f, z_0) = \lim_{z \to z_0} (z - z_0) f(z)$. |
| **Residue Calculation (Pole of Order n):** | If $f(z)$ has a pole of order n at $z_0$, then $\text{Res}(f, z_0) = \frac{1}{(n-1)!} \lim_{z \to z_0} \frac{d^{n-1}}{dz^{n-1}} [(z - z_0)^n f(z)]$. |

## Applications of Residue Theorem

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| **Improper Integrals:** | The Residue Theorem can be used to evaluate improper integrals of the form $\int_{-\infty}^{\infty} f(x) dx$. |
| **Trigonometric Integrals:** | The Residue Theorem can be used to evaluate integrals of the form $\int_0^{2\pi} F(\cos \theta, \sin \theta) d\theta$. |