CHEAT SHEETS HERO

Category Theory

A comprehensive cheat sheet covering monoidal category theory, its prerequisites, string diagrams, and compact closed categories. Useful for students and researchers in mathematics, physics, and computer science.

Prerequisites: Category Theory Basics

Categories

Definition: A category C consists of:

- A class of objects, denoted ob(C).
- For each pair of objects A, B, a set of morphisms (or arrows) Hom(A, B).
- A composition law: for $f: A \to B$ and $g: B \to C$, there exists a morphism $g \circ f: A \to C$.
- For each object A, an identity morphism id$A$: $A \to A$.

Axioms:

- Associativity: $h \circ (g \circ f) = (h \circ g) \circ f$
- Identity: $f \circ \mathrm{id}A = f = \mathrm{id}B \circ f$ for $f: A \to B$

Examples:

- **Set**: Sets as objects, functions as morphisms.
- **Vect**: Vector spaces as objects, linear transformations as morphisms.
- **Grp**: Groups as objects, group homomorphisms as morphisms.

Functors: A functor $F: C \to D$ maps objects and morphisms from category C to category D preserving composition and identity.

Isomorphisms: A morphism $f: A \to B$ is an isomorphism if there exists a morphism $g: B \to A$ such that $g \circ f = \mathrm{id}A$ and $f \circ g = \mathrm{id}B$.

Natural Transformations: A natural transformation $\alpha: F \to G$ between functors $F, G: C \to D$ assigns to each object A in C a morphism $\alpha A: F(A) \to G(A)$ in D such that for any morphism $f: A \to B$ in C, $G(f) \circ \alpha A = \alpha B \circ F(f)$.

Products and Coproducts

Product: The product of objects A and B in a category C is an object $A \times B$ together with morphisms $\pi_1: A \times B \to A$ and $\pi_2: A \times B \to B$ such that for any object X and morphisms $f: X \to A$ and $g: X \to B$, there exists a unique morphism $h: X \to A \times B$ such that $\pi_1 \circ h = f$ and $\pi_2 \circ h = g$.

Coproduct: The coproduct of objects A and B in a category C is an object $A + B$ together with morphisms $\iota_1: A \to A + B$ and $\iota_2: B \to A + B$ such that for any object X and morphisms $f: A \to X$ and $g: B \to X$, there exists a unique morphism $h: A + B \to X$ such that $h \circ \iota_1 = f$ and $h \circ \iota_2 = g$.

Examples (Set):

- Product: Cartesian product of sets.
- Coproduct: Disjoint union of sets.

Examples (Vect):

- Product: Direct product of vector spaces.
- Coproduct: Direct sum of vector spaces.

Monoidal Categories

Monoidal Category Structure

Definition: A monoidal category $(C, \otimes, I, \alpha, \lambda, \rho)$ consists of:

- A category C.
- A bifunctor $\otimes: C \times C \to C$ (tensor product).
- An object $I \in \mathrm{ob}(C)$ (unit object).
- A natural isomorphism $\alpha: (A \otimes B) \otimes C \to A \otimes (B \otimes C)$ (associator).
- Natural isomorphisms $\lambda: I \otimes A \to A$ and $\rho: A \otimes I \to A$ (left and right unitors).

Axioms (Coherence):

- Pentagon identity: A coherence condition for the associator.
- Triangle identity: A coherence condition relating the associator, left unitor, and right unitor.

Examples of Monoidal Categories

- **(Set, ×, {∗})**: Sets with Cartesian product and a singleton set as the unit.
- **(Vect, ⊗, K)**: Vector spaces over a field K with the tensor product and K itself as the unit.
- **(Cat, ×, 1)**: Categories with the Cartesian product and the terminal category as the unit.

Braided Monoidal Categories

Definition: A braided monoidal category is a monoidal category C with a natural isomorphism $\beta: A \otimes B \to B \otimes A$ (braiding) satisfying additional coherence axioms.

Examples:

- Symmetric monoidal categories where β is an involution ($\beta^2 = \mathrm{id}$).
- The category of representations of a quantum group.

String Diagrams

Basic Conventions

- Objects are represented by wires.
- Morphisms are represented by boxes on the wires.
- Composition is represented by connecting boxes vertically.
- Tensor product is represented by placing wires side by side horizontally.

String Diagrams for Monoidal Categories

- Associator α: $(A \otimes B) \otimes C \to A \otimes (B \otimes C)$ is a crossing.
- Left unitor λ: $I \otimes A \to A$ is a bending to the left.
- Right unitor ρ: $A \otimes I \to A$ is a bending to the right.
- Identity id: $A \to A$ is a straight wire.

Examples

String diagrams provide a visual representation that makes complex compositions and tensor products easier to understand. For example, a diagram showing the composition of multiple morphisms in a monoidal category will clearly show the flow of data and how different parts of the diagram interact.

Compact Closed Categories

Definition and Properties

Definition: A compact closed category is a symmetric monoidal category where every object A has a dual object A^{**} and morphisms ηA: $I \to A \otimes A^{**}$ (unit) and εA: $A^{**} \otimes A \to I$ (counit) satisfying the snake equations.

Snake Equations:

$(\varepsilon A \otimes \mathrm{id}A) \circ (\mathrm{id}A^{**} \otimes \eta A) = \mathrm{id}A^{**}$

$(\mathrm{id}A \otimes \varepsilon A^{**}) \circ (\eta A \otimes \mathrm{id}A) = \mathrm{id}A$

Graphical Representation

- The unit ηA is represented by a cup.
- The counit εA is represented by a cap.
- The snake equations are depicted as straightening a bent wire.

Examples

- **Hilb**: Hilbert spaces with bounded linear operators.
- **Rel**: Sets and relations.